Int. J. Solids Structures, 1975, Vol. 11, pp. 841-852. Pergamon Press. Printed in Great Britain.

THE PROPAGATION OF STRESS WAVES INTO A LAMINATED HALF SPACE USING A SECOND ORDER MICROSTRUCTURE THEORY†

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(Received 29 July 1974)

Abstract—A second order microstructure theory for the dynamic behaviour of elastic laminates is applied to the problem of a laminated half space, with interfaces normal to the boundary, subjected to harmonically time varying displacement and stress distributions at the boundary. The finite number of modes of the microstructure theory is found to be sufficient to model a uniform normal displacement boundary condition but not a uniform normal stress boundary condition. The solutions yield the constituent displacement and stress distributions both near the boundary and in the far field and permit an assessment of the usefulness of the microstructure theory for such boundary value problems.

1. INTRODUCTION

In recent years, continuum theories with microstructure for the analysis of dynamic processes in composite materials have been developed, first by Sun, Herrmann and Achenbach [1–3] and more recently by Hegemier, Gurtman and Nayfeh [4], in which the individual deformational behaviours of the composite constituents appear explicitly. These theories have shown impressive success in predicting the dispersion of steady state plane waves [1–13] and determining the transient material response near the wave front for comparison with impact experiments [4, 10–13].

Using a second order formulation‡ of the Sun, Herrmann and Achenbach theory for application to elastic laminates [14], the authors in [15] presented a comparison of the displacement and stress distributions associated with the steady state wave propagation modes with the corresponding distributions obtained using the theory of elasticity for plane waves propagating normal to, parallel to, and at an angle to the laminate interfaces. It was found that over a significant range of frequency the stress and displacement distributions of the second order theory closely matched the distributions predicted by the theory of elasticity.

In this paper, the second order theory is applied to the problem of a laminated half space, with layer interfaces normal to the boundary, which is subjected to harmonically time varying displacement and stress distributions at the boundary. The study of this problem is motivated by several questions of research interest. First, is the finite modal content of the microstructure theory sufficient for modeling typical boundary conditions met in practice? If so, what stress and displacement distributions result, both near the boundary and in the far field, and how do they vary with frequency? Further, solutions of this type are of practical importance to those involved with engineering applications of composite materials. Finally, they are of research interest as an example of the application of a theory with microstructure.

The theory used in this paper is a member of the family comprised of coupled stress theories, multipolar theories and micromorphic material theories developed by Mindlin [16], Green, Naghdi and Rivlin [17] and Eringen and Suhubi [18], respectively. Heterogeneous materials,

†This research was supported by the United States Atomic Energy Commission and by National Science Foundation Grant No. GK-35055.

‡The order is here taken to mean the number of terms retained in addition to the zero-order term in the expansion of the layer displacement vectors in terms of distance normal to the layer.

particularly laminates, constitute the simplest test case for such theories. Thus the present work clarifies and further demonstrates the applicability of a microstructural theory to a class of real materials.

2. FORMULATION

The development will rely closely on the presentation of the second order theory in [14, 15] and will use the same notation. The geometry considered is a half space of an elastic bilaminate with layer interfaces normal to the half space boundary as shown in Fig. 1. The bilaminate consists of alternating layers of a matrix material, with Lamé constants λ_m, μ_m, density ρ_m and layer thickness d_m, and a reinforcing material, with Lamé constants λ_f, μ_f, density ρ_f and layer thickness d_f.

The problem considered is the response of the half space to two boundary conditions, a uniform, harmonically time varying, normal displacement boundary condition, and a uniformly distributed, harmonically time varying, normal stress boundary condition. These conditions give rise to waves propagating in the x_1 direction which are symmetric about the layer centerlines.

The second order theory of [14, 15] involves writing the displacement vector components of the kth reinforcing layer u_i^{fk} as the expansion

$$u_i^{fk} = u_{0i}^{fk} + x_2^f \psi_i^{fk} + \frac{1}{2}(3x_2^{f_2} - d_f^2)\Phi_i^{fk}, \tag{1}$$

where x_2^f is the x_2 distance from the layer center line. In the development of the theory, the discrete layer variables u_{0i}^{kf}, ψ_i^{fk}, Φ_i^{fk} are approximated by continuous variables u_{0i}^f, ψ_i^f, Φ_i^f which, together with corresponding variables u_{0i}^m, ψ_i^m, Φ_i^m for the matrix material, are the microstructural variables of the theory.

For symmetric waves propagating parallel to the layers, symmetry reduces the expansion equation (1) to

$$u_1^{fk} = u_{01}^{fk} + \frac{1}{2}(3x_2^{f^2} - d_f^2)\Phi_1^{fk} \tag{2}$$

$$u_2^{fk} = x_2^f \psi_2^{fk}, \tag{3}$$

so that the problem has six microstructural variables: u_{01}^f, u_{01}^m, ψ_2^f, ψ_2^m, Φ_1^f, Φ_1^m. The plane strain equations of motion for this case appear as equations (26–35) of [14]. Substituting steady state wave solutions

$$u_{01}^f = \bar{u}_{01}^f e^i(\omega t - kx_1) \tag{4}$$

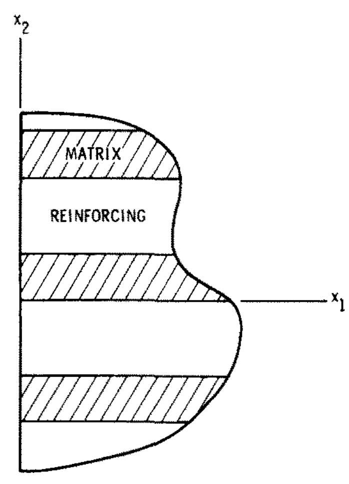

Fig. 1. Laminated half space.

$$u_{01}^{m} = \bar{u}_{01}^{m} e^{i}(\omega t - kx_1) \tag{5}$$

$$\psi_2^{f} = \bar{\psi}_2^{f} e^{i}(\omega t - kx_1) \tag{6}$$

$$\vdots$$

where t is time, ω is frequency and k is wave number, into equations (26–35) of [14] yields a set of homogeneous equations. The vanishing of the determinant of the coefficients of these equations determines k for a given ω, and the null vector of the equations yields the modal components $\bar{u}_{01j}^{f}$, $\bar{u}_{01j}^{m}$, ... corresponding to the jth root k_j. Substituting the modal component back into the expansion, equations (2) and (3), then determines the displacement distributions for that mode and from them the stress distributions can be obtained.

This determination of the modal displacement and stress distributions was carried out for the lowest modes in [14] and [15]. In this paper, the objective is to superimpose the second-order modal distributions to attempt to match specified boundary conditions at the half space boundary.

a. *Displacement boundary condition*

For the uniform displacement boundary condition case, the exact boundary condition at $x_1 = 0$ is

$$u_1^{fk} = u_1^{mk} = De^{i\omega t}, \tag{7}$$

where D is a constant. No condition is placed on the x_2 components u_2^{fk} and u_2^{mk} or on the shear stress at the boundary. In terms of the modal distributions

$$\bar{u}_{1j}^{f} = \bar{u}_{01j}^{f} + \frac{1}{2}(3x_2^{f^2} - d_f^2)\bar{\Phi}_{1j}^{f} \tag{8}$$

$$\bar{u}_{1j}^{m} = \bar{u}_{01j}^{m} + \frac{1}{2}(3x_2^{m^2} - d_m^2)\bar{\Phi}_{1j}^{m}, \tag{9}$$

the boundary conditions at $x_1 = 0$ can be written approximately as

$$\sum_j q_j \bar{u}_{1j}^{f} \cong D \tag{10}$$

$$\sum_j q_j \bar{u}_{1j}^{m} \cong D \tag{11}$$

where q_j are constants chosen to satisfy the boundary conditions, equations (10) and (11), as closely as possible in some sense. In this work a simple least squares criterion has been chosen, minimizing the integral

$$I(q_j) = \int_{-(d_f/2)}^{d_f/2} \left(\sum_j q_j \bar{u}_{1j}^{f} - D\right)\left(\sum_j q_j \bar{u}_{1j}^{f} - D\right)^* dx_2^{f}$$
$$+ \int_{-(d_m/2)}^{d_m/2} \left(\sum_j q_j \bar{u}_{1j}^{m} - D\right)\left(\sum_j q_j \bar{u}_{1j}^{m} - D\right)^* dx_2^{m} \tag{12}$$

where the superscript * denotes the complex conjugate. The minimization, with equations (8) and (9), yields a set of linear equations for the q_i

$$\sum_j a_{kj} q_j^* = b_k \tag{13}$$

where

$$b_k = \left(d_f\bar{u}^f_{01k} - \frac{3}{8}d_f^3\bar{\Phi}^f_{1k} + d_m\bar{u}^m_{01k} - \frac{3}{8}d_m^3\bar{\Phi}^m_{1k}\right)D \tag{14}$$

and

$$a_{kj} = d_f\bar{u}^f_{01k}(\bar{u}^f_{01j})^* - \frac{3}{8}d_f^3[\bar{u}^f_{01k}(\bar{\Phi}^f_{1j})^* + \bar{\Phi}^f_{1k}(\bar{u}^f_{01j})^*]$$
$$+\frac{49}{320}d_f^5\bar{\Phi}^f_{1k}(\bar{\Phi}^f_{1j})^* + d_m\bar{u}^m_{01k}(\bar{u}^m_{01j})^* - \frac{3}{8}d_m^3[\bar{u}^m_{01k}(\bar{\Phi}^m_{1j})^* + \bar{\Phi}^m_{1k}(\bar{u}^m_{01j})^*] + \frac{49}{320}d_m^5\bar{\Phi}^m_{1k}(\bar{\Phi}^m_{1j})^*. \tag{15}$$

Once the q_j are determined, the solutions are obtained by summing the modal components.

$$u^f_{01} = \sum_j q_j\bar{u}^f_{01j} \tag{16}$$

$$u^m_{01} = \sum_j q_j\bar{u}^m_{01j} \tag{17}$$

$$\vdots$$

b. *Stress boundary condition*

The strain components in the kth reinforcing layer are obtained from equations (2) and (3) using the usual strain-displacement relations

$$\epsilon^{fk}_{11} = \frac{\partial u_1^{fk}}{\partial x_1} = \frac{\partial u^{fk}_{01}}{\partial x_1} + \frac{1}{2}(3x_2^{f2} - d_f^2)\frac{\partial \Phi_1^{fk}}{\partial x_1} \tag{18}$$

$$\epsilon^{fk}_{22} = \frac{\partial u_2^{fk}}{\partial x_2} = \psi_2^{fk} \tag{19}$$

$$\epsilon^{fk}_{12} = \frac{1}{2}\left(\frac{\partial u_1^{fk}}{\partial x_2} + \frac{\partial u_2^{fk}}{\partial x_1}\right) = \frac{1}{2}\left(3x_2^f\Phi_1^{fk} + x_2^f\frac{\partial \psi_2^{fk}}{\partial x_1}\right). \tag{20}$$

The corresponding stress components are then obtained from the usual linear isotropic relation

$$t^{fk}_{ij} = \lambda_f\delta_{ij}\epsilon^{fk}_{mm} + 2\mu_f\epsilon^{fk}_{ij}. \tag{21}$$

By using equations (18–20) and the solutions (4)... in (21), the modal normal stress component $\bar{t}^f_{11j}$ is

$$\bar{t}^f_{11j} = -ik_j(\lambda_f + 2\mu_f)\left[\bar{u}^f_{01j} - \frac{1}{2}(3x_2^{f2} - d_f^2)\bar{\Phi}^f_{1j}\right] + \lambda_f\bar{\psi}^f_{2j} \tag{22}$$

with a corresponding equation for the matrix material.

For a uniformly distributed normal stress distribution, the boundary conditions in terms of superimposed modal distributions are

$$\sum_j q_j\bar{t}^f_{11j} \cong T \tag{23}$$

$$\sum_j q_j\bar{t}^m_{11j} \cong T \tag{24}$$

where T is the constant amplitude of the time-harmonic stress at $x_1 = 0$. Again no conditions are placed on the tangential displacements or shear stresses at the boundary.

Minimizing the integral

$$I(q_i) = \int_{-(d_f/2)}^{d_f/2} \left(\sum_j q_j \bar{t}^f_{11j} - T\right)\left(\sum_j q_j \bar{t}^f_{11j} - T\right)^* \mathrm{d}x_2^f + \int_{-(d_m/2)}^{d_m/2} \left(\sum_j q_j \bar{t}^m_{11j} - T\right)\left(\sum_j q_j \bar{t}^m_{11j} - T\right)^* \mathrm{d}x_2^m \quad (25)$$

then yields a linear equation for the q_i in terms of the modal components as in the displacement case.

3. RESULTS

The characteristic equation of the second order theory for plane-strain symmetric waves propagating parallel to the layers was found to have three roots k_j for a given ω, yielding a total of three modes of propagation. One root (referred to hereafter as mode 1) is real for all values of ω. A second one (mode 2) is imaginary until a particular cut-off frequency is reached, after which it is real. The third (mode 3) is complex for the range of frequencies examined.

In Fig. 2, the real and imaginary parts of k_j are plotted vs frequency for the three modes.† Mode 1 propagates without attenuation for all ω. Mode 2 is a standing wave below the cut-off frequency; above it, it propagates without attenuation. Mode 3 propagates but is attenuated for all ω. In Fig. 3, the phase velocity ω/Real (k_j) is plotted vs frequency for the three modes.

In Fig. 4, the u_1^f and u_1^m displacement distributions (normalized to one at the matrix material centerline) are shown for the three modes at $\omega = 6$ Mrad/s (0·955 MHz). The corresponding values of k_j are 0·639, $-4{\cdot}53i$ and $1{\cdot}06$–$5{\cdot}38i$ 1/mm, respectively.

The three lowest modes (in terms of the smallest absolute value of k_j) obtained using the theory of elasticity for the same frequency have k_j values of 0·639, $-5{\cdot}78i$ and $2{\cdot}28$–$4{\cdot}11i$. The u_1^f and u_1^m distributions for these three modes are shown in Fig. 5. Although the mode 1 distributions of the two theories are identical, the mode 2 and 3 distributions are qualitatively different for the second order microstructure theory compared to the theory of elasticity. The two theories are different, and there is no reason to expect that they would invariably produce the same mode shapes. Still, as will be demonstrated by the results, it is an interesting revelation that

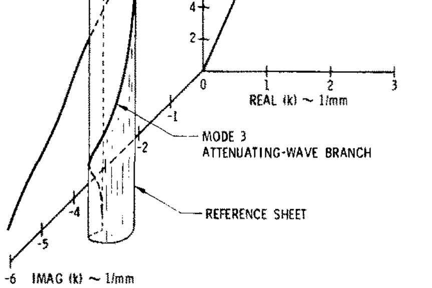

Fig. 2. Modal dispersion curves.

†Results have been computed with the material properties used in [15]: for the reinforcing material, $\lambda_f = 50$ GPa, $\mu_f = 33{\cdot}33$ GPa, $\rho_f = 1$ Mg/m^3, $d_f = 1$ mm; for the matrix material, $\lambda_m = 0{\cdot}7777$ GPa, $\mu_m = 0{\cdot}3333$ GPa, $\rho_m = 0{\cdot}3333$ Mg/m^3, $d_m = 0{\cdot}25$ mm. In terms of the notation introduced in [1], these parameters correspond to $\gamma = \mu_f/\mu_m = 100$, $\theta = \rho_f/\rho_m = 3$, $\eta = d_f/(d_f + d_m) = 0{\cdot}8$, $\nu_f = \lambda_f/2(\lambda_f + \mu_f) = 0{\cdot}3$, $\nu_m = \lambda_m/2(\lambda_m + \mu_m) = 0{\cdot}35$, $\xi = kd_f = k$, and $\beta = \omega/\text{Real}\,(k)/\sqrt{(\mu_m/\rho_m)} = \omega/\text{Real}\,(k)$.

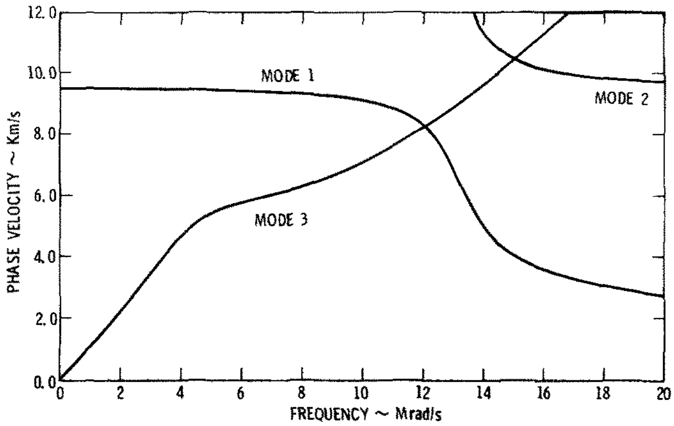

Fig. 3. Phase velocity curves.

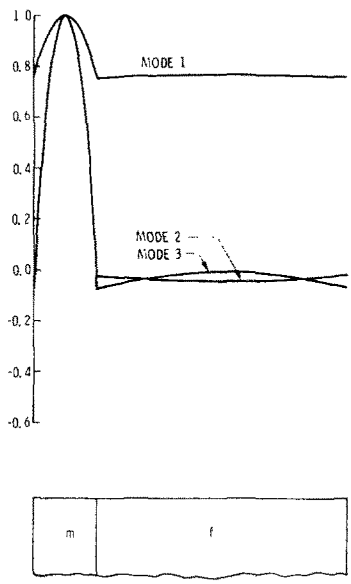

Fig. 4. Microstructure theory displacement mode shapes.

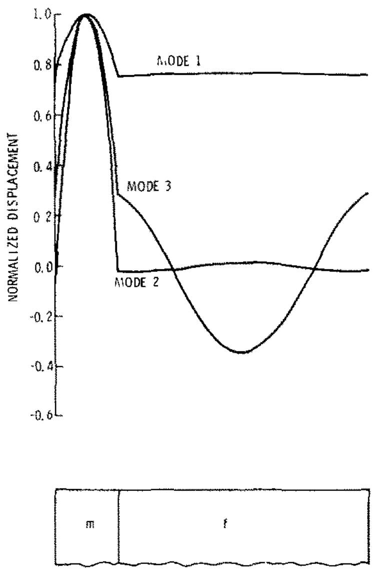

Fig. 5. Elasticity theory displacement mode shapes.

the boundary conditions are accurately modeled by the microstructure theory even though the mode shapes are dissimilar.

The solution for the uniform displacement boundary condition case was obtained using a unit amplitude displacement at the boundary ($D = 1$ mm). When the q_j were determined and the solutions determined by summing the modal distributions, it was found that the solution exactly matched the boundary condition: $u_1^f = u_1^m = D$. In Fig. 6, the propagated displacement distributions (far from the boundary) are shown for $\omega = 1, 7, 13$ and 20 Mrad/s (0·159, 1·11, 2·07 and 3·18 MHz). At $\omega = 1$ and 7 Mrad/s, only mode 1 propagates without attenuation. Above the cut-off frequency ($\omega = 12{\cdot}8$ Mrad/s, 2·04 MHz), both modes 1 and 2 propagate without attenuation. The propagated displacement distributions are seen to differ markedly from the uniform boundary distribution. If Fig. 7, the average propagated displacement in each material is shown vs frequency for this case. Here it is clearly seen that the cut-off frequency is a resonant condition at which the displacement in the matrix material becomes large compared to the boundary displacement while the reinforcing material displacement decreases.

In Fig. 8, the normal stress distributions at the boundary are shown for various frequencies for the displacement boundary condition, and in Fig. 9, the corresponding propagated

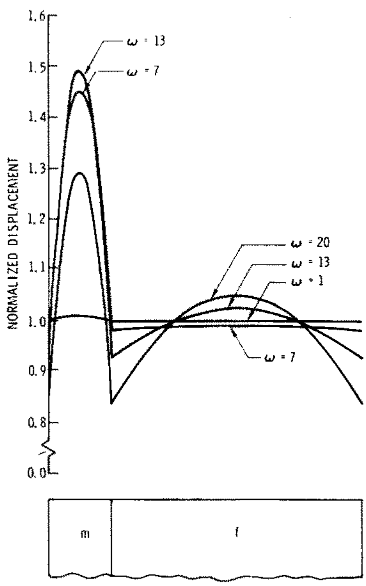

Fig. 6. Propagated displacement distributions.

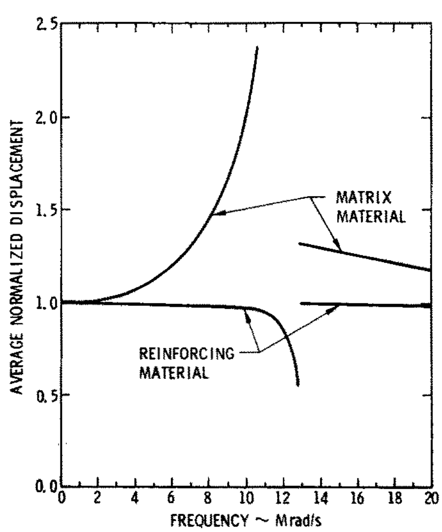

Fig. 7. Propagated average displacements.

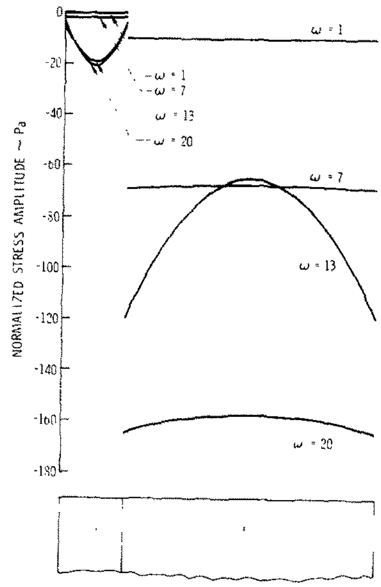

Fig. 8. Boundary stress distributions.

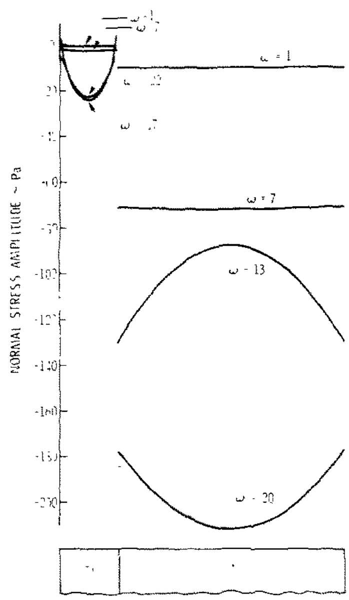

Fig. 9. Propagated stress distributions.

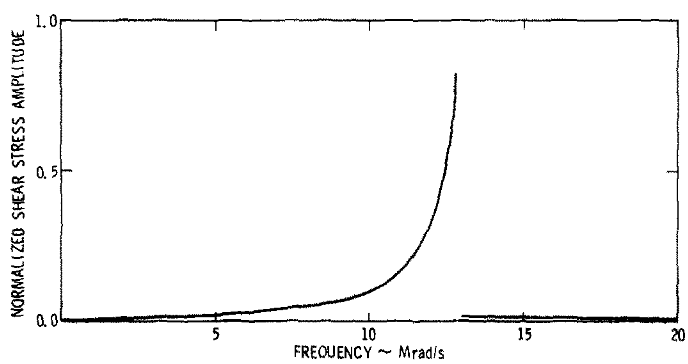

Fig. 10. Propagated interface shear stress.

distributions are shown. An essential point is that the propagated stresses can be substantially larger than the stresses at the boundary. In Fig. 10, the propagated interface shear stress between the layers, normalized by the average normal stress at the boundary, is plotted vs frequency for this case. Again the cut-off frequency is seen to be a critical point where the interface shear stress becomes of the same order as the normal stress at the boundary.

Finally, in Fig. 11 the energy distribution among the modes in the propagated wave is shown vs frequency. Below the cut-off frequency, only mode 1 propagates. Above the cut-off, the energy is approximately equally divided between modes 1 and 2, with mode 2 carrying progressively more of the energy as frequency increases.

Next, the q_i were determined for the case of a unit amplitude stress at the boundary ($T = 1$ Pa). In this case, it was found that the modal content of the microstructure theory was not sufficient to approximate a uniform stress distribution at the boundary. The actual normal stress distribution at the boundary obtained is shown in Fig. 12. Although the stress distribution in the matrix material is non-uniform, the average stress in each layer is approximately one.

In Fig. 13 the propagated normal stress distributions are shown for this case. Note that the propagated stresses are substantially larger than the stresses imposed at the boundary. In Fig. 14, the average propagated normal stress in each material is shown vs frequency for this case. Note the large reinforcing material stress at frequencies below the cut-off. Above the cut-off, the stress

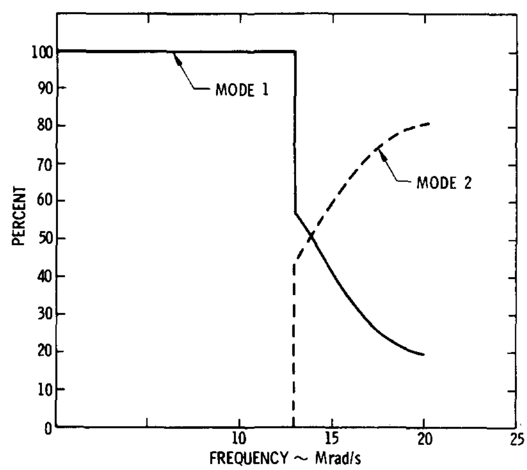

Fig. 11. Modal energy distribution.

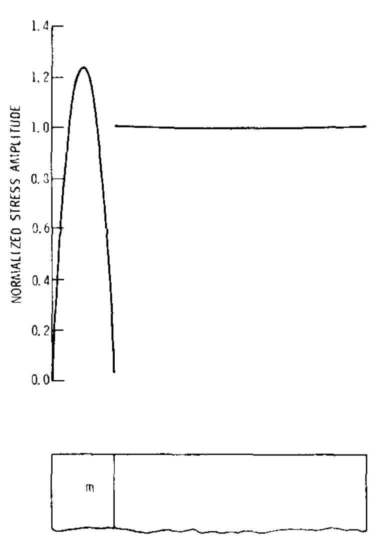

Fig. 12. Boundary stress distribution.

approaches one in both materials. In Fig. 15 the propagated interface shear stress is shown vs frequency. At the cut-off, it is over three times the imposed boundary stress.

In Fig. 16 the energy distribution among the modes is shown vs frequency for this case. At the cut-off frequency, the energy abruptly changes from mode 1 to mode 2, then rapidly goes predominantly back to mode 1 above the cut-off frequency.

4. SUMMARY AND CONCLUSIONS

A second order microstructure theory has been applied to a laminated half space subjected to harmonic uniform displacement and uniform stress boundary conditions. The finite number of modes in the theory was found sufficient to solve the uniform displacement case, and the resulting propagated stress and displacement distributions were shown.

On the other hand, it was found that a uniform stress boundary condition could not be achieved, although the average normal stress in each constituent was equal to the imposed boundary stress.

A very significant finding was that propagated stress levels can exceed by several times the stresses applied at the boundary.

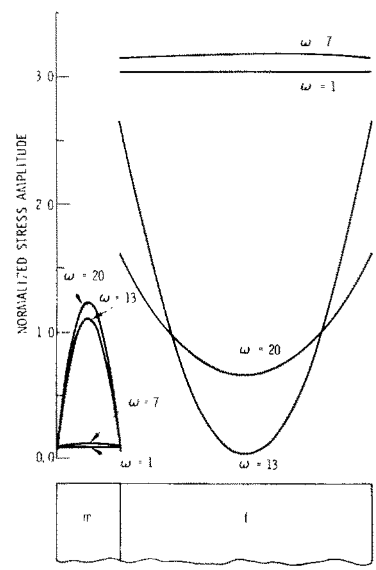

Fig. 13. Propagated stress distributions.

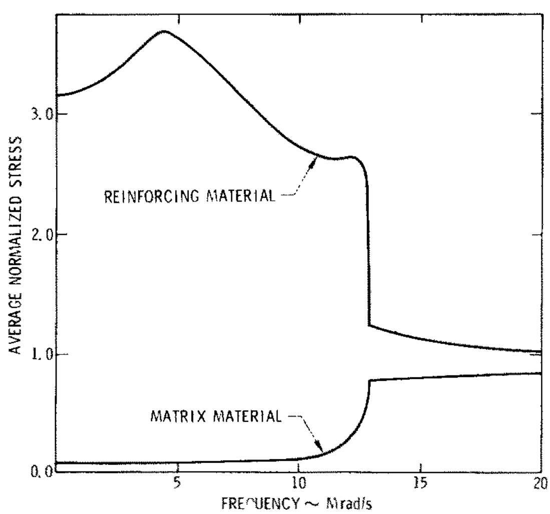

Fig. 14. Propagated average stresses.

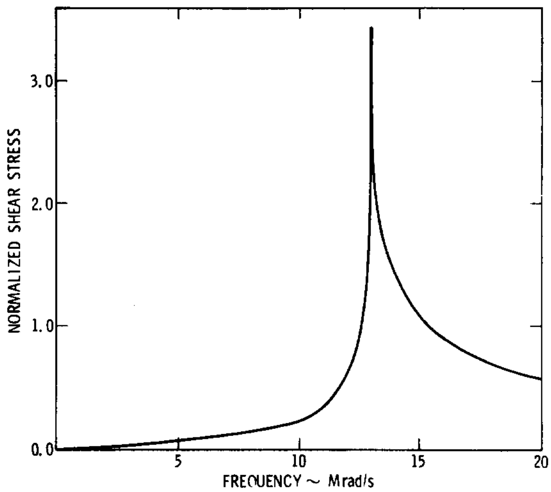

Fig. 15. Propagated interface shear stress.

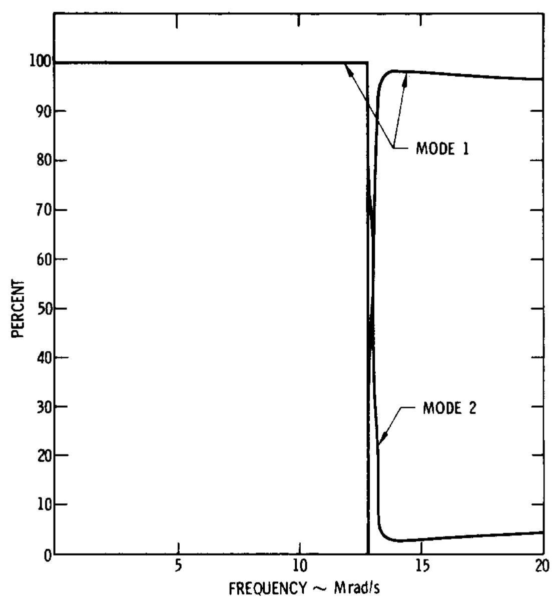

Fig. 16. Modal energy distribution.

Boundary conditions on the displacements parallel to the boundary and the shear stresses at the boundary were not imposed in obtaining the results presented. Since the average displacement parallel to the boundary and the average shear stress at the boundary in each layer vanish for the type of waves considered, the effect on the propagated results should be slight. However, when an attempt was made to include shear stress boundary conditions in the stress case, the result was a degradation of the achieved normal stress distribution. As a result of this, and the fact that even without the shear stress conditions a uniform stress boundary condition could not be achieved, it is clear that, except for a very restricted class of boundary conditions, obtaining accurate stress and displacement solutions near the boundary in inhomogeneous materials will require a theory of higher order than the one we have considered.

REFERENCES

1. C.-T. Sun, J. D. Achenbach and G. Herrmann, Continuum theory for a laminated medium. *J. Appl. Mech.* **35**, 467 (1968).
2. J. D. Achenbach and G. Herrmann, Effective stiffness theory for a laminated composite. *Proc. 10th Midwestern Mech. Conf.* pp. 91–106. Johnson Publishing Company (1968).
3. J. D. Achenbach and C.-T. Sun, The directionally reinforced composite as a homogeneous continuum with microstructure. *Dynamics of Composite Materials* pp. 48–69. American Society of Mechanical Engineers (1972).

4. G. A. Hegemier, On a theory of interacting continua for wave propagation in composites. *Dynamics of Composite Materials* pp. 70–121. American Society of Mechanical Engineers (1972).
5. C.-T. Sun, J. D. Achenbach and G. Herrmann, On the vibration of a laminated body. *J. Appl. Mech.* **35**, 689 (1968).
6. J. D. Achenbach and G. Herrmann, Wave motion in solids with lamellar structuring. *Dynamics of Structural Solids* p. 23. American Society of Mechanical Engineers (1968).
7. G. Herrmann and J. D. Achenbach, Applications of theories of generalized cosserat continua to the dynamics of composite materials. *Mechanics of Generalized Continua* pp. 69–79. Springer, Berlin (1968).
8. G. Herrmann and J. D. Achenbach, Wave propagation in laminated and fiber-reinforced composites. *Mechanics of Composite Materials* pp. 337–360. Pergamon Press, Oxford (1970).
9. C. Sve, Time harmonic waves travelling obliquely in a periodically laminated medium. *J. Appl. Mech.* **38**, 477 (1971).
10. G. A. Hegemier and A. H. Nayfeh, A continuum theory for wave propagation in laminated composites. Case 1: Progagation normal to the laminates. *J. Appl. Mech.* **40**, 503 (1973).
11. G. A. Hegemier and T. C. Bache, A continuum theory for wave propagation in laminated composites. Case 2: Propagation parallel to the laminates. *J. Elasticity* **3**, 125 (1973).
12. C. Sve and J. S. Whittier, One-dimensional pulse propagation in an obliquely laminated half space. *J. Appl. Mech.* **37**, 778 (1970).
13. C. Sve and S. Okubo, Experiments on pulse propagation in an obliquely laminated composite. Air Force Report No. SAMSO-TR-74-74, Aerospace Corporation (1974).
14. D. S. Drumheller and A. Bedford, On a continuum theory for a laminated medium. *J. Appl. Mech.* **40**, 527 (1973).
15. D. S. Drumheller and A. Bedford, Wave propagation in elastic laminates using a second order microstructure theory. *Int. J. Solids Struct.* **10**, 61 (1974).
16. R. D. Mindlin, Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.* **16**, 51 (1964).
17. A. E. Green, P. M. Naghdi and R. S. Rivlin, Directors and multipolar displacements in continuum mechanics. *Intl. J. Engr. Sci.* **2**, 611 (1965).
18. A. C. Eringen and E. S. Suhubi, Nonlinear theory of simple microelastic solids—I. *Intl. J. Engr. Sci.* **2**, 189 (1964).